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# Reflection and refraction of thermal waves from a surface or an interface between dissimilar materials

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**Abstract**—The harmonic analysis is applied to study the reflection and refraction patterns of thermal waves from a surface and an interface between dissimilar materials. Regardless of the boundary conditions imposed on the surface/interface, the reflection angle is found to be identical to the incident angle of thermal waves. The refraction angle of the thermal waves penetrating into the second material layer, however, depends on the ratio of the thermal wave speeds in the two material layers. The relative strengths of reflection and refraction waves to that of the incident wave are obtained for all the cases. The physical envelope for complete transmission and the physical conditions for complete reflection are discussed in detail. The results thus obtained are extended to study the reflection and refraction patterns of thermal shock waves in the neighborhood of a material interface. The relative thermal Mach numbers are used to characterize the reflection and refraction behavior of thermal shock waves.

## INTRODUCTION

THE THERMAL wave theory describes a delayed response between the heat flux vector and the temperature gradient in the process of heat conduction. It is a high-rate model specially developed for the short time behavior of thermal disturbances propagating in solids. In contrast to the classical diffusion theory, the wave model depicts a finite speed of heat propagation which becomes important when the time-rate change of temperature is high. A quantitative criterion for the dominance of the wave nature over the diffusion behaviour has been derived [1]:

$$T_{,t} \gg \exp\left[(C^2/\alpha)t\right]. \qquad (1)$$

It reveals that the wave nature in heat conduction can be important if (i) the relaxation time $\tau = \alpha/C^2$ [2, 3] is large, (ii) the response time $t$ is short, and (iii) the rate of change of temperature $T_{,t}$ is high. While the first condition deals with the intrinsic properties of the solid medium, the second and the third conditions also involve the thermal loading and local geometry of the system. A localized moving heat source with high intensity [4, 5], a rapidly propagating crack tip [6, 7] in solids, and an interface between dissimilar materials [8, 9], for example, are known situations where the wave behavior is pronounced. In addition to the existence of a sharp wavefront [9–13] in the propagation of thermal waves, the thermal resonance phenomena [14, 15] and the shock wave formation are characteristics pertinent to the wave theory which cannot be depicted by the classical diffusion model. When extended to the prediction of crack initiations around a fast-moving heat source [16], the wave theory depicts a crack initiation angle depending on the thermal wave speed. Also, a wider damage zone behind the heat source was found as a result of the thermal shock formation. In these studies, the diffusion model assuming an infinite speed for heat propagation appears as a special case. Generalization from the diffusion model to the wave theory involves much more than a switch from a parabolic to a hyperbolic equation. Rigorous considerations for the physical basis of the thermal wave theory include the extension of Gibb's equation for the thermodynamic irreversibility [17, 18], kinetic theory of molecular collisions [19, 20], and causality restrictions imposed by the special theory of relativity [21, 22]. Along with the past research on the thermal wave propagation, a complete and detailed survey has been made in a recent annual review article by the author [1]. It classifies the past research according to their individual emphases.

The recent work by Zehnder and Rosakis [23] sheds light on the experimental verification for the thermal shock formation around a rapidly propagating crack tip. For a crack propagating in a 4340 steel with a speed of 900 m $s^{-1}$, the transonic solution [6, 24] of the temperature field in the near-tip region preserves several salient features observed in the experiment. They include (i) the isotherm patterns parallel to the crack surface, (ii) the normal shock formation as a result of the intensified thermal energy cumulated in the immediate vicinity ahead of the crack tip, and (iii) a constant temperature gradient in the near-tip region. Should the classical diffusion model be used to describe the corresponding phenomena, a family of parabolic isotherms and a $1/\sqrt{r}$-type of singularity for the temperature gradient in the near-tip region result [24]. Besides, the intensification of thermal

## NOMENCLATURE

| | |
|---|---|
| $A$ | amplitude of thermal waves [m] |
| $c$ | phase velocity of thermal waves [m $s^{-1}$] |
| $C$ | thermal wave speed [m $s^{-1}$] |
| $\mathbf{i}_1, \mathbf{i}_2$ | unit vectors in the $x_1$- and $x_2$-directions |
| $k$ | thermal wave number [number of waves $m^{-1}$] |
| $\mathbf{n}$ | unit vector in the direction of thermal wave propagation |
| $q$ | heat flux vector [W $m^{-2}$] |
| $Q_1, Q_2$ | amplitudes of heat flux waves in the $x_1$ and $x_2$ directions |
| $r$ | position vector [m] |
| $R_x$ | ratio of $X^{(I)}/X^{(II)}$ |
| $M$ | thermal Mach number defined as $v/C$ |
| $t$ | physical time [s] |
| $T$ | temperature [°C] |
| $v$ | velocity of moving source [m $s^{-1}$] |
| $x_1, x_2$ | physical coordinates with origin at the reflection point [m]. |

Greek symbols

| | |
|---|---|
| $\alpha$ | thermal diffusivity [$m^2$ $s^{-1}$] |
| $\gamma$ | damping rate [$s^{-1}$] |
| $\eta$ | integral variable |
| $\theta$ | direction angle of thermal waves measured from the vertical ($x_2$) axis [deg] |
| $\kappa$ | thermal conductivity [W $m^{-1}$ °$C^{-1}$] |
| $\tau$ | relaxation time [s] |
| $\omega$ | frequency of thermal waves [$s^{-1}$]. |

Subscripts and superscripts

| | |
|---|---|
| $\mathbf{X}$ | vector quantity of $X$ |
| $X_i$ | $i = 1, 2$: the component of vector $\mathbf{X}$ in the $x_i$ direction |
| $X_{,j}$ | $j \equiv t, 1, 2$: derivative of $X$ with respect to $j$ |
| $X_{(i)}$ | $i = 0, 1, 2$: kinematic or thermodynamic quantity $X$ in the incident ($i = 0$), reflected ($i = 1$), and refracted ($i = 2$) waves |
| $X^{(j)}$ | $j \equiv$ I, II: thermodynamic quantities $X$ in the material layer ($j$). |

energy ahead of the crack tip is much weaker. These phenomena are extremely important for the detailed understanding of the energy release rate around a dynamically propagating crack tip in solids. Due to preservation of this unusual behavior in the temperature field, future development of the thermal wave theory seems promising.

The present paper develops fundamental understanding of the reflection and refraction of thermal waves by a surface and an interface between dissimilar materials. Along with the thermal shock formation [4–7] and the thermal resonance under frequency excitations [14, 15], it is another feature in the thermal wave propagation. For the case involving a free surface, the phase response of the incident and the reflected waves is studied under various boundary conditions. For the case involving a material interface, on the other hand, the emphasis is placed on the refraction behavior in relation to the relative thermal wave speeds. The physical conditions for the complete reflection and transmission of thermal waves will be obtained by the relative strength of reflected and refracted waves. In some situations, in addition to the damping characteristics in time, the transmitted wave decays in the principal direction of propagation and the wave nature in refraction diminishes. Lastly, an extension of the results will be made for the thermal shock waves induced by a moving source in the supersonic range. The thermal Mach numbers are used to characterize the reflection and refraction of thermal waves.

## TEMPERATURE AND FLUX WAVES

In the absence of a heat source, the energy equation for the thermal wave propagation can either have a temperature ($T$) or a heat flux ($\mathbf{q}$) representation [4–7, 9–12]:

$$\nabla^2 T = (1/C^2)T_{,tt} + (1/\alpha)T_{,t} \quad (T\text{-representation})$$

$$\nabla(\nabla \cdot \mathbf{q}) = (1/C^2)\mathbf{q}_{,tt} + (1/\alpha)\mathbf{q}_{,t} \quad (\mathbf{q}\text{-representation}) \quad (2)$$

with $C$ and $\alpha$ being respectively the thermal wave speed and the thermal diffusivity of the solid medium. Note that due to the phase lag between the temperature gradient and the heat flux vector, the constitutive equation in the linearized thermal wave theory is

$$\mathbf{q}(\mathbf{r}, t) = -(C^2\kappa/\alpha)\exp(-C^2 t/\alpha) \times \int^t \nabla T(\mathbf{r}, \eta)\exp(C^2\eta/\alpha)\,d\eta \quad (3)$$

where $\kappa$ is the thermal conductivity and the ratio of $\alpha/C^2$ is defined as the relaxation time $\tau$ [2]. Due to such a complicated integral relationship, it has been shown that the $\mathbf{q}$-formulation is more convenient to use for boundary value problems involving flux-specified conditions [5, 7, 9, 12].

Unlike the stress or acoustic waves, the thermal wave possesses a damping behavior resulting from the thermal diffusivity. The relaxation distance defined as $C\tau$ dictates the transition of thermal waves from an

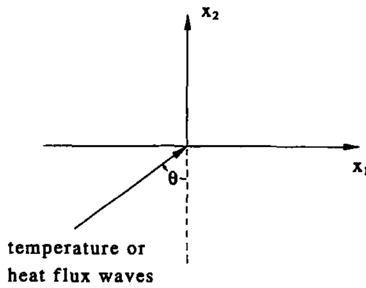


FIG. 1. Propagation of temperature and heat-flux waves in solids and the coordinate system.

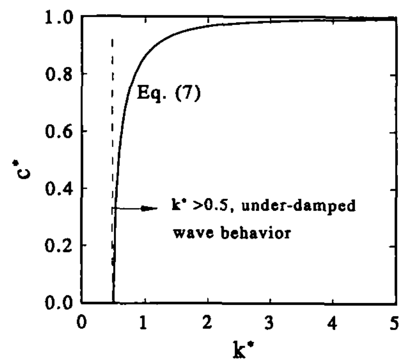


FIG. 2. The dispersion relation between $k^*$ and $c^*$ for thermal waves. $k^*$ must be greater than 1/2 for an under-damped wave behavior.

over-damped to an under-damped behavior [14, 15]. For temperature waves propagating in solids, the damping behavior suggests the following wave form:

$$T(\mathbf{r}, t) = A \exp(\gamma t) \exp[ik(\mathbf{n}\cdot\mathbf{r} - ct)] \quad (4)$$

where $\gamma$ is the rate of damping, $c$ the phase velocity, $k$ the wave number, and $\mathbf{n}$ the unit vector in the direction of thermal wave propagation. For a two-dimensional wave shown in Fig. 1, $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2$ and $\mathbf{n} = \sin\theta\mathbf{i}_1 + \cos\theta\mathbf{i}_2$. Equation (4) thus becomes

$$T(x_1, x_2, t) = A \exp(\gamma t) \exp[ik(x_1 \sin\theta + x_2 \cos\theta - ct)]. \quad (5)$$

In the absence of thermal damping, the value of $\gamma$ is zero and the phase velocity ($c$) is identical to the wave speed $C$. Equation (5) in this case reduces to the form of displacement waves propagating in an elastic solid [25]. In the presence of thermal damping, the damping rate ($\gamma$) and the phase velocity ($c$) can be determined by substituting equation (5) into the $T$-representation in equation (2). It results

$$\gamma = -1/2\tau = -C^2/2\alpha \quad (6)$$

and

$$c/C = \sqrt{(1 - (C/2\alpha k)^2)}. \quad (7)$$

Note that the wave frequency defined as $\omega(k) = ck = C[k^2 - (C/2\alpha)^2]^{1/2}$ renders that $\mathrm{d}^2\omega/\mathrm{d}k^2 \neq 0$. According to the definition for physical waves [26], therefore, the thermal wave is dispersive in nature. This is a salient feature distinguishing the thermal waves from the stress or acoustic waves. Moreover, it is noted that the phase velocity must be real. This condition requires that, for $\alpha$, $k$, and $C$ being all positive

$$k > C/2\alpha. \quad (8)$$

In the presence of damping in general, equation (8) gives the physical condition for an under-damped behavior to exist in the thermal wave propagation. For the wave number $k$ being smaller than the critical value of $C/2\alpha$, as demonstrated alternatively in terms of the wave frequency and modal number [14, 15], the diffusion behavior dominates over the wave behavior and an over-damped wave presence. Since no physical wave would exist in this case, the present analysis will be confined to the wave number restricted by equation (8). In terms of the dimensionless wave number $k^* = k/(C/\alpha)$ and phase speed $c^* = c/C$, the dispersion relation of the temperature wave, namely equation (7), is displayed in Fig. 2.

The propagating form of the heat flux wave can be determined in the same fashion. In two dimensions, the $\mathbf{q}$-representation of the energy equation has the following components:

$$q_{1,11} + q_{2,12} = (1/C^2)q_{1,tt} + (1/\alpha)q_{1,t}$$
$$q_{2,12} + q_{2,22} = (1/C^2)q_{2,tt} + (1/\alpha)q_{2,t} \quad (9)$$

where $q_1$ and $q_2$, respectively, are the components of the heat flux vector $\mathbf{q}$ in the $x_1$ and $x_2$ directions and the subscripts denote differentiations. After careful manipulations, it can be shown that the heat flux waves possess the same under-damped behavior as the temperature wave. The corresponding expressions to equation (5) are

$$q_1(x_1, x_2, t) = Q_1 \exp(-t/2\tau) \exp[ik(x_1 \sin\theta + x_2 \cos\theta - ct)]$$
$$q_2(x_1, x_2, t) = Q_2 \exp(-t/2\tau) \exp[ik(x_1 \sin\theta + x_2 \cos\theta - ct)], \quad \text{with} \quad Q_2 = Q_1 \cot\theta \quad (10)$$

where $c$ is the phase velocity obtained in equation (7). The relative strength of $q_2$ to $q_1$ is $\cot\theta$. In the case of $\theta = 90°$, refer to Fig. 1, the heat flux vector propagates along the $x_1$-direction. The value of $\cot\theta$ is zero in this case and consequently $q_2 = 0$. For $\theta = 0°$, the heat flux vector propagates along the $x_2$-direction and the amplitude of $Q_1$ must be zero in order to maintain a finite value of $q_2$. Consequently, $q_1 = 0$ in this case.

In passing, note that the function $\exp(-t/2\tau)$ approaches zero for the diffusion model assuming $C$ being infinity. The wave forms represented by equations (5) and (10), therefore, do not exist for a diffusion behavior in heat conduction. For an extremely short time behavior with $1/C^2 \gg 1/\alpha$, on the other hand, $\tau$ approaches infinity and equations (5) and (10) reduce to the standard forms of waves without damping.

**REFLECTION FROM A SURFACE**

Since both temperature and flux waves possess the same under-damped behavior, reflection of thermal waves can be understood by studying the temperature wave alone. Referring to Fig. 3, let us consider an infinite surface subjecting to a temperature-specified boundary condition:

$$T = 0 \quad \text{at} \quad x_2 = 0. \tag{11}$$

The origin of the coordinate system $(x_1, x_2)$ is assigned at the reflection point. The temperature wave contains two components. The incident component enters from the direction of $\theta_{(0)}$ measured from the vertical axis. The wave train of the incident wave is thus expressed by

$$T_{(0)}(x_1, x_2, t) = A_{(0)} \exp(-t/2\tau) \times \exp[ik_{(0)}(x_1 \sin\theta_{(0)} + x_2 \cos\theta_{(0)} - c_{(0)}t)]. \tag{12}$$

The reflected wave, on the other hand, departs from the surface along the direction of $\theta_{(1)}$. The wave train of the reflected wave is therefore

$$T_{(1)}(x_1, x_2, t) = A_{(1)} \exp(-t/2\tau) \times \exp[ik_{(1)}(x_1 \sin\theta_{(1)} - x_2 \cos\theta_{(1)} - c_{(1)}t)]. \tag{13}$$

In these equations, subscripts (0) and (1) denote respectively the kinematic quantities in the incident and reflected waves. The phase velocities $c_{(0)}$ and $c_{(1)}$, for example, are

$$c_{(0)} = \sqrt{(1-(C/2\alpha k_{(0)})^2)}C \quad \text{and} \quad c_{(1)} = \sqrt{(1-(C/2\alpha k_{(1)})^2)}C. \tag{14}$$

At the surface where the temperature is zero, combination of the two components gives

$$T = T_{(0)} + T_{(1)} = A_{(0)} \exp[ik_{(0)}(x_1 \sin\theta_{(0)} - c_{(0)}t)] + A_{(1)} \exp[ik_{(1)}(x_1 \sin\theta_{(1)} - c_{(1)}t)] = 0 \quad \text{at} \quad x_2 = 0. \tag{15}$$

Since this equation holds for any values of $x_1$ and $t$, it implies

$$k_{(0)} \sin\theta_{(0)} = k_{(1)} \sin\theta_{(1)}, \quad k_{(0)}c_{(0)} = k_{(1)}c_{(1)}, \quad \text{and} \quad A_{(1)} = -A_{(0)}. \tag{16}$$

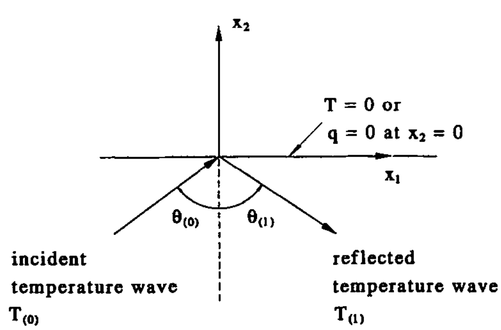


FIG. 3. The incident and reflected temperature waves from a surface subjecting to a temperature- or flux-specified boundary condition.

With the assistance of equation (14), moreover, equation (16) leads to

$$k_{(1)} = k_{(0)}, \quad \theta_{(1)} = \theta_{(0)}, \quad c_{(1)} = c_{(0)}, \quad \text{and} \quad A_{(1)} = -A_{(0)}. \tag{17}$$

This result demonstrates that (i) the reflected angle of the thermal wave is identical to the incident angle and (ii) the temperature-specified boundary condition does not alter the wave number and phase velocity after reflections. By substituting equation (17) into equation (13), moreover, it yields

$$T_{(1)}(x_1, x_2, t) = A_{(0)} \exp(-t/2\tau) \times \exp[ik_{(0)}(x_1 \sin\theta_{(0)} + x_2 \cos\theta_{(0)} - c_{(0)}t) \pm i\pi]. \tag{18}$$

In comparison with the incident wave, equation (12), a phase shift of $\pi$ after reflection is evident.

A similar approach can be taken should the surface be subjected to a flux-specified boundary condition:

$$q_2 = 0 \quad \text{at} \quad x_2 = 0. \tag{19}$$

In this case, refer to equation (10), superposition of the incident and the reflected flux waves at $x_2 = 0$ gives

$$[Q_{1(0)} \cot\theta_{(0)}] \exp[ik_{(0)}(x_1 \sin\theta_{(0)} - c_{(0)}t)] + [Q_{1(1)} \cot\theta_{(1)}] \exp[ik_{(1)}(x_{(1)} \sin\theta_{(1)} - c_{(1)}t)] = 0. \tag{20}$$

For all values of $x_1$ and $t$, equation (20) holds if

$$\theta_{(1)} = \theta_{(0)}, \quad k_{(1)} = k_{(0)}, \quad c_{(1)} = c_{(0)}, \quad Q_{1(1)} = -Q_{1(0)}, \quad \text{and consequently} \quad Q_{2(1)} = -Q_{2(0)}. \tag{21}$$

For both components $Q_1$ and $Q_2$, again, a phase shift of 180° results after reflections. Note that the incident and reflected temperature waves resulting from equation (10), by the decomposition of equation (3), are

$$T_0(x_1, x_2, t) = \frac{Q_{1(0)}}{\kappa \sin\theta_{(0)}} \exp(-t/2\tau) \times \left\{\tau c_{(0)} \exp[ik_{(0)}(x_1 \sin\theta_{(0)} + x_2 \cos\theta_{(0)} - c_{(0)}t)] + \frac{1}{2k_{(0)}} \exp\{i[k_{(0)}(x_1 \sin\theta_{(0)} + x_2 \cos\theta_{(0)} - c_{(0)}t) + \pi/2]\}\right\} \tag{22}$$

$$T_1(x_1, x_2, t) = \frac{Q_{1(0)}}{\kappa \sin\theta_{(0)}} \exp(-t/2\tau) \times \left\{\tau c_{(0)} \exp[ik_{(0)}(x_1 \sin\theta_{(0)} - x_2 \cos\theta_{(0)} - c_{(0)}t)] + \frac{1}{2k_{(0)}} \exp\{i[k_{(0)}(x_1 \sin\theta_{(0)} - x_2 \cos\theta_{(0)} - c_{(0)}t) + 3\pi/2]\}\right\}. \tag{23}$$

At the surface of $x_2 = 0$, clearly, the incident and reflected waves are out-of-phase by 180°. In contrast to the previous case with temperature-specified boundary condition, however, additional components with phase shifts being respectively $\tau/2$ and $3\tau/2$ exist in the incident and reflected waves. In the thermal wave propagation, the phase shift strongly depends on the boundary conditions imposed on the surface. Should the boundary condition (11) be replaced by $T_{,2} = 0$ at $x_2 = 0$, for example, it can be shown that the incident and reflected waves become *in phase* and no phase shift exists after reflection.

## REFLECTION AND REFRACTION FROM A MATERIAL INTERFACE

A more complicated situation results when the thermal wave penetrates through an interface between dissimilar materials. Denoting the kinematic quantities in the refracted wave by subscript (2), as shown in Fig. 4, superscripts (I) and (II) are used to denote the thermal properties in each material layer. The temperature will be used in deriving the refracted angle $\theta_{(2)}$ in relation to the incident angle $\theta_{(0)}$. The continuities for both temperature and heat flux waves

$$T^{(I)} = T^{(II)} \quad \text{and} \quad q_2^{(I)} = q_2^{(II)} \quad \text{at} \quad x_2 = 0 \quad (24)$$

are imposed as the boundary conditions across the material interface.

For one-dimensional waves, Frankel *et al.* [9] showed that continuity of heat flux in equation (24) results in a condition containing mixed-derivatives of temperature with respect to space and time. The flux formulation, therefore, was employed instead. Also, when the relaxation time in the two material layers are equal, i.e. $\tau^{(I)} = \tau^{(II)}$, they found that the condition of flux continuity in equation (24) reduces to that in Fourier's law of heat conduction, i.e. $\kappa^{(I)}T^{(I)}_{,2} = \kappa^{(II)}T^{(II)}_{,2}$. Analytical solutions exist for this case which were used to validate the numerical solutions employing the Runge–Kutta method for $\tau^{(I)} \neq \tau^{(II)}$. By examining their solutions for both temperature and heat flux distributions across the material interface, however, the effect of different ratios of $\tau^{(I)}/\tau^{(II)}$ is insignificant. Larger values of $\tau^{(II)}$ only brings the transmitted ripple closer to the material interface while the wave behavior in the neighborhood of the interface remains unchanged.

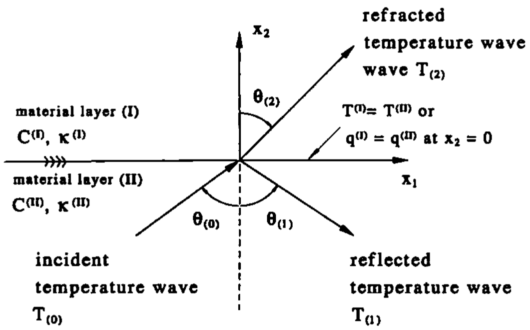


FIG. 4. Reflection and refraction of temperature waves from the interface between material layers (I) and (II).

The same forms of temperature waves are allowed in each material layer:

$$T^{(II)}_{(0)} = A_{(0)} \exp(-t/2\tau^{(II)}) \times \exp[ik_{(0)}(x_1 \sin\theta_{(0)} + x_2 \cos\theta_{(0)} - c^{(II)}_{(0)}t)] \quad \text{for the incident wave}$$

$$T^{(II)}_{(1)} = A_{(1)} \exp(-t/2\tau^{(II)}) \times \exp[ik_{(1)}(x_1 \sin\theta_{(1)} - x_2 \cos\theta_{(1)} - c^{(II)}_{(1)}t)] \quad \text{for the reflected wave}$$

$$T^{(I)}_{(2)} = A_{(2)} \exp(-t/2\tau^{(I)}) \times \exp[ik_{(2)}(x_1 \sin\theta_{(2)} + x_2 \cos\theta_{(2)} - c^{(I)}_{(2)}t)] \quad \text{for the refracted wave} \quad (25)$$

where

$$c^{(j)}_{(i)} = C^{(j)}\sqrt{(1 - (C^{(j)}/2\alpha^{(j)}k_{(i)})^2)} \quad \text{for} \quad i = 0, 1, 2 \quad \text{and} \quad j = \text{I, II}. \quad (26)$$

The same constraint as equation (8)

$$k_{(0,1)} > C^{(II)}/2\alpha^{(II)} \quad \text{and} \quad k_{(2)} > C^{(I)}/2\alpha^{(I)} \quad (27)$$

has also to be satisfied for under-damped waves to exist in each material layer. In terms of the incident, reflected, and refracted waves, the continuities of temperature and flux waves at the material interface of $x_2 = 0$ gives

$$A_{(0)} \exp[ik_{(0)}x_1 \sin\theta_{(0)}] \exp[-(ik_{(0)}c^{(II)}_{(0)} + 1/2\tau^{(II)})t] + A_{(1)} \exp[ik_{(1)}x_1 \sin\theta_{(1)}] \times \exp[-(ik_{(1)}c^{(II)}_{(1)} + 1/2\tau^{(II)})t] = A_{(2)} \exp(-t/2\tau^{(I)}) \times \exp[ik_{(2)}x_1 \sin\theta_{(2)}] \exp[-(ik_{(2)}c^{(I)}_{(2)} + 1/2\tau^{(I)})t]. \quad (28)$$

For any values of $x_1$ and $t$, equation (28) is satisfied if

$$k_{(0)} \sin\theta_{(0)} = k_{(1)} \sin\theta_{(1)} = k_{(2)} \sin\theta_{(2)},$$
$$k_{(0)}c^{(II)}_{(0)} = k_{(1)}c^{(II)}_{(1)} = k_{(2)}c^{(I)}_{(2)},$$
$$\tau^{(I)} = \tau^{(II)}. \quad (29)$$

With the assistance of equation (26), as shown previously, equation (29) is equivalent to

$$k_{(1)} = k_{(0)}, \quad \theta_{(1)} = \theta_{(0)} \quad \text{for reflection waves}$$
$$k_{(2)} = [C^{(II)}/C^{(I)}]k_{(0)},$$
$$\sin\theta_{(2)} = [C^{(I)}/C^{(II)}] \sin\theta_{(0)} \quad \text{for refraction waves.} \quad (30)$$

Note that the condition of $\tau^{(I)} = \tau^{(II)}$, i.e. $C^{(I)}/C^{(II)} = \sqrt{(\alpha^{(I)}/\alpha^{(II)})}$, in equation (29) seems very restrictive for *harmonic waves with damping* to exist in both sides of the material interface. The functional approach employing the finite Fourier transform [9] or Green's function [27] allows more general forms of waves which do not require such a strong condition. It has been observed, however, that discontinuity of relaxation time from one material layer to another only affects the refracted wave forms while the interfacial behavior of thermal waves remains the same. This is especially true for $\tau^{(I)} < \tau^{(II)}$ according to the work of Frankel *et al.* [9]. With regard to the behavior of the refracted wave relative to the incident wave in the neighborhood of the material interface, therefore, the harmonic analysis employed in this work is sufficient although a strong constraint is imposed through the continuity of relaxation times.

By equation (30), the continuities of temperature and heat flux waves across the material interface renders

$$A_{(0)} + A_{(1)} = A_{(2)}$$

$$A_{(0)} - A_{(1)} = \frac{\kappa^{(I)}}{\kappa^{(II)}} \cdot \frac{C^{(II)}}{C^{(I)}} \cdot \frac{\cos\theta_{(2)}}{\cos\theta_{(0)}} A_{(2)}. \quad (31)$$

The relative strength of the reflected and the refracted wave to the incident wave can thus be obtained:

$$\frac{A_{(1)}}{A_{(0)}} = \frac{R_C \cos\theta_{(0)} - R_\kappa \sqrt{(1 - R_C^2 \sin^2\theta_{(0)})}}{R_C \cos\theta_{(0)} + R_\kappa \sqrt{(1 - R_C^2 \sin^2\theta_{(0)})}}$$

$$\frac{A_{(2)}}{A_{(0)}} = \frac{2R_C \cos\theta_{(0)}}{R_C \cos\theta_{(0)} + R_\kappa \sqrt{(1 - R_C^2 \sin^2\theta_{(0)})}} \quad (32)$$

where $R_X = X^{(I)}/X^{(II)}$ with $X \equiv \kappa$, $C$ and the relations in equation (30) have been used. In the presence of a material interface, the reflected angle of the thermal wave is still the same as the incident angle, i.e. $\theta_{(1)} = \theta_{(0)}$. While the refracted angle ($\theta_{(2)}$) depends only on the ratio of the thermal wave speed ($R_C$), as shown by the relation of $\sin\theta_{(2)} = R_C \sin\theta_{(0)}$ in equation (30), the relative strengths of reflected ($A_{(1)}/A_{(0)}$) and refracted ($A_{(2)}/A_{(0)}$) waves also depend on the ratio of thermal conductivities ($R_\kappa$). The following cases are important for the reflection and refraction of thermal waves:

(a) *Complete reflection.* Note that equation (32) makes physical sense only if $\sin\theta_{(2)} = R_C \sin\theta_{(0)} < 1$. In the case that $R_C \sin\theta_{(0)} > 1$, $\cos\theta_{(2)} = i[(R_C \times \sin\theta_{(0)})^2 - 1]^{1/2}$ which is *purely imaginary*. The refracted wave in this case, refer to equation (25), becomes

$$T_{(2)}^{(I)} = A_{(2)} \exp\{-k_{(2)}[(R_C \sin\theta_{(0)})^2 - 1]^{1/2} x_2\}$$

$$\times \exp(-t/2\tau^{(I)}) \exp[ik_{(2)}(x_1 \sin\theta_{(2)} - c_{(2)}^{(I)} t)]. \quad (33)$$

While an oscillatory wave behavior is still present in the $x_1$-direction, the refracted wave decays exponentially in the direction of $x_2$. With regard to the relative strength of the reflected wave $A_{(1)}/A_{(0)}$ in equation (32), most importantly, the numerator and denominator appear as complex conjugate in this case. It reveals that *no refraction* would occur because the amplitude, and hence the wave energy proportional to the square of the amplitude, of the incident and the reflected waves are equal. A phase shift, however, results between the incident and the reflected waves. It is only in the limiting case that $R_C \sin\theta_{(0)} = 1$, $A_{(1)} = A_{(0)}$ and the incident and the reflected waves become in phase. In equation (33), due to the absence of a wave behavior in the $x_2$-direction, 'refraction' angle $\theta_{(2)}$ does not possess a physical meaning. This phenomenon has also been observed for displacement and stress waves penetrating through an interface between elastic media [25].

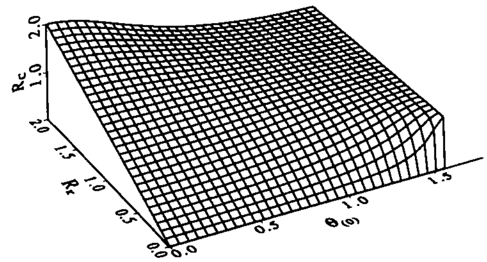

FIG. 5. The envelope for the complete transmission of temperature waves, equation (33).

(b) *Complete transmission.* Another important situation is the complete transmission of the temperature wave into the material layer (I). For an incident wave with a finite amplitude $A_{(0)}$, this situation is described by a zero-strength of the *reflected* wave, i.e. $A_{(1)}/A_{(0)}$. It results

$$\cos^2\theta_{(0)} + (R_\kappa \sin\theta_{(0)})^2 = (R_\kappa/R_C)^2. \quad (34)$$

Figure 5 shows the envelope of complete transmission in terms of the ratios of $R_\kappa$, $R_C$, and the incident angle $\theta_{(0)}$. When $\theta_{(0)} = \pi/2$, the case of a normal incident wave, complete transmission may occur only if $R_C = 1$ as depicted by equation (34). For smaller values of $R_\kappa$, refer to the planes with $R_\kappa$ being constant, the value of $R_C$ decreases with the incident angle $\theta_{(0)}$ for complete transmission. For larger values of $R_\kappa$, instead, the value of $R_C$ increases when the value of $\theta_{(0)}$ decreases. For $R_C = 0.5$, as an example, Figs. 6 and 7 display the relative amplitudes of reflected ($A_{(1)}/A_{(0)}$) and refracted ($A_{(2)}/A_{(0)}$) waves represented by equation (32). The flat surface in Fig. 6 represents the situations for

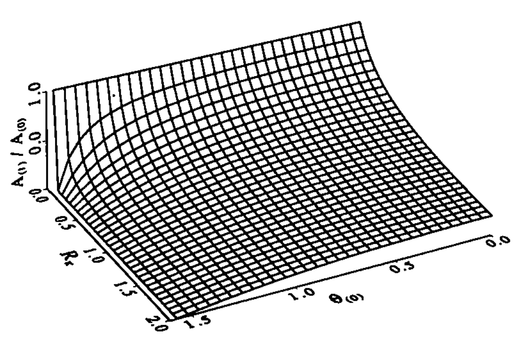

FIG. 6. The relative strength of reflection wave, $A_{(1)}/A_{(0)}$ in equation (31) with $R_C = 0.5$.

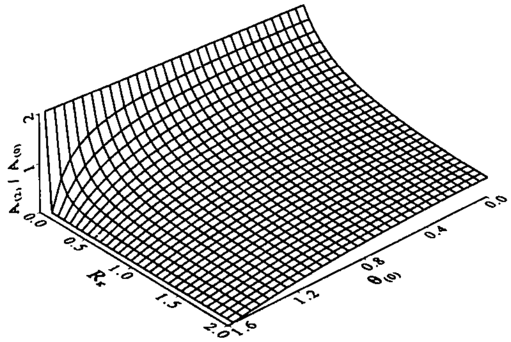

FIG. 7. The relative strength of refraction wave, $A_{(2)}/A_{(0)}$ in equation (31) with $R_C = 0.5$.

complete transmission and no reflection wave exists. Note also that limiting values of $A_{(1)}/A_{(0)}$ and $A_{(2)}/A_{(0)}$ being 1 and 2, respectively, exist at $\theta_{(0)} = 0^\circ$ when $R_\kappa$ approaches zero.

## REFLECTION AND REFRACTION OF THERMAL SHOCK WAVES

For a heat source [4, 5] or a crack tip [6, 7] propagating at a speed faster than that of the heat propagation in a solid, shock waves will be formed around the moving origin. The thermal shock wave thus formed reflects the insufficient time for dissipating heat into the surrounding media and the thermal energy thus cumulates at the shock surface. For a heat source moving in the material layer (II) with a velocity $v$, as illustrated in Fig. 8, the shock wave is oriented in a direction of $\sin^{-1}(1/M^{(II)})$ measured from the trailing edge of the heat source. The thermal Mach number $M^{(II)}$ is defined as $v/C^{(II)}$. For the thermal shock formation, the speed of the moving source $v$ must be greater than or equal to the thermal wave speed $C^{(II)}$ and consequently $M^{(II)} \geqslant 1$.

Reflection and refraction of thermal shock waves are important because they may result in delamination or interfacial cracking under extreme conditions [1, 16]. For the thermal shock coming from the material layer (II) with an incident angle $\theta_{(0)} = \pi/2 - \sin^{-1}(1/M^{(II)})$, refer to equation (30), it yields

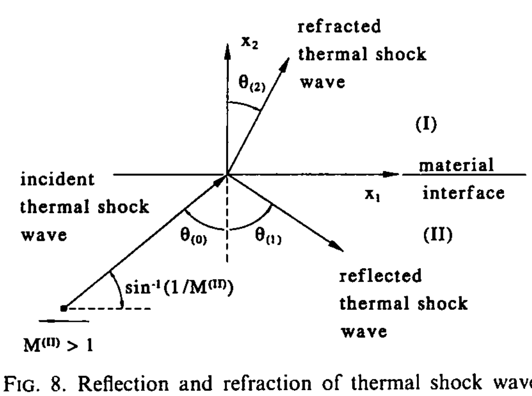


FIG. 8. Reflection and refraction of thermal shock waves induced by a fast-moving source in the material layer (II), $M^{(II)} > 1$.

$$\theta_{(1)} = \pi/2 - \sin^{-1}(1/M^{(II)})$$

for the reflection wave, and

$$\sin\theta_{(2)} = (M^{(II)}/M^{(I)})\sin\theta_{(0)} = [M^{(II)2} - 1]^{1/2}/M^{(I)}$$

for the reflection wave (35)

where $M^{(I)} = v/C^{(I)}$ is the thermal Mach number relative to the thermal wave speed $C^{(I)}$ in the material layer (I). In equation (35), the result of $\sin\theta_{(0)} = [M^{(II)2} - 1]^{1/2}/M^{(II)}$ has been used. For refraction waves to exist physically, the condition of $\sin\theta_{(2)} \leqslant 1$ must be satisfied which requires

$$M^{(II)2} - M^{(I)2} < 1. \tag{36}$$

The physical significance of equation (36) will be discussed later. For the thermal Mach numbers $M^{(I)}$ and $M^{(II)}$ in this range, equation (35) depicts that

$$\sin\theta_{(2)}/\sin\theta_{(0)} = M^{(II)}/M^{(I)} = C^{(I)}/C^{(II)}. \tag{37}$$

Because the sine function is monotonically increasing in the range from 0 to $\pi/2$, equation (37) implies

$$\theta_{(2)} > \theta_{(0)} \text{ if } C^{(I)} > C^{(II)}, \quad \theta_{(2)} < \theta_{(0)} \text{ if } C^{(I)} < C^{(II)}. \tag{38}$$

This situation is further illustrated in Fig. 9. The refracted shock wave sweeps toward the material interface if the thermal wave speed $C^{(I)}$ is larger than $C^{(II)}$. It departs from the material interface, on the other hand, if the thermal wave speed $C^{(I)}$ is smaller than $C^{(II)}$. With these special values of $\theta_{(0)}$ and $\theta_{(2)}$, the relative amplitudes of reflected and refracted waves shown by equation (32) become

$$\frac{A_{(1)}}{A_{(0)}} = \frac{R_C M^{(I)} - R_\kappa M^{(II)}\sqrt{(M^{(I)2} - M^{(II)2} + 1)}}{R_C M^{(I)} + R_\kappa M^{(II)}\sqrt{(M^{(I)2} - M^{(II)2} + 1)}}$$

$$\frac{A_{(2)}}{A_{(0)}} = \frac{2R_C M^{(I)}}{R_C M^{(I)} + R_\kappa M^{(II)}\sqrt{(M^{(I)2} - M^{(II)2} + 1)}}. \tag{39}$$

The condition for complete transmission, equation (34), reduces to

$$\frac{M^{(I)}}{M^{(II)}} = \left(\frac{R_\kappa}{R_C}\right)\sqrt{(M^{(I)2} - M^{(II)2} + 1)} \tag{40}$$

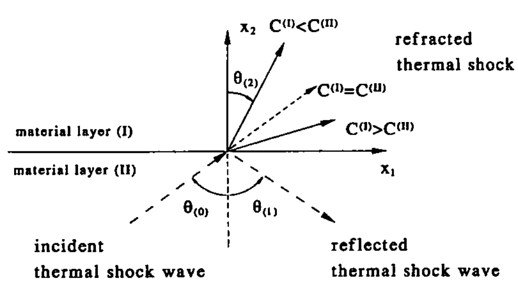


FIG. 9. Two possible paths of wave refraction for $C^{(I)} > C^{(II)}$ and $C^{(I)} < C^{(II)}$. The path for $C^{(I)} = C^{(II)}$ is in line with the incident wave.

in terms of the thermal Mach numbers and the ratio of $R_\kappa/R_C$.

The physical meaning of the constraint shown by equation (36) is now clear. For $M^{(II)2}-M^{(I)2} > 1$, the quantity $M^{(I)2}-M^{(II)2}+1$ inside the square roots in equations (39) and (40) is negative. The numerator and denominator of $A_{(1)}/A_{(0)}$ in equation (39) thus become complex conjugate. In this case, as discussed in section (a) in the previous section, the incident and reflected waves have the same amplitude and wave energy, i.e. $|A_{(1)}| = |A_{(0)}|$, and no refraction wave could exist. Similar to equation (33), the transmitted wave into the material layer (I) has the form of

$$T^{(I)}_{(2)} = A_{(2)} \exp[-k_{(2)}x_2(M^{(II)2} - M^{(I)2}-1)^{1/2}/M^{(I)}] \exp(-t/2\tau^{(I)}) \cdot \exp[ik_{(2)}x_1(M^{(II)2}-1)^{1/2}/M^{(I)} - c^{(I)}_{(2)}t] \quad (41)$$

which decays in the $x_2$-direction. Equation (36), therefore, provides the physical condition for the presence of a complete wave behavior on *both* sides of the material interface. Should the conditions of $R_C \sin\theta_{(0)} > 1$ (refer to equation (33)) or equation (41) be violated, the transmitted wave into the material layer (I) decays in the $x_2$-direction. Consequently, refraction of temperature *waves* makes no physical sense.

Under the constraint of equation (36), the refracted path of the shock wave shown in Fig. 9 needs to be re-examined. Retrieving $M^{(I)} = v/C^{(I)}$ and $M^{(II)} = v/C^{(II)}$, equation (36) becomes

$$v < \frac{C^{(I)}C^{(II)}}{\sqrt{(C^{(I)2}-C^{(II)2})}}. \quad (42)$$

Since the velocity of the moving source must be real, first of all, the wave speed $C^{(I)}$ must be greater than $C^{(II)}$. The path departing from the material interface for $C^{(I)} < C^{(II)}$ in Fig. 9, therefore, is not acceptable for a wave behavior in refraction. For $C^{(I)} > C^{(II)}$, the thermal energy is carried away from the interface faster than that incident upon it. No thermal energy will cumulate at the interface and the refracted wave inclined toward the material interface. Secondly, an upper bound being $C^{(I)}C^{(II)}/[C^{(I)2}-C^{(II)2}]^{1/2}$ exists for the speed of the moving source. Should the speed $v$ exceed this bound value, a decayed behavior in the $x_2$-direction is present and the wave nature in refraction disappears.

Graphically, the constraint shown by equation (36) is displayed in Fig. 10. Since $M^{(II)}$ must be greater than one for the thermal shock formation, refraction waves can only exist in the dotted area between the hyperbola ($M^{(II)2}-M^{(I)2} = 1$) and its asymptote ($M^{(II)} = M^{(I)}$). Within the same domain of $M^{(I)}$ and $M^{(II)}$, Fig. 11 shows the surface represented by equation (40) for the complete transmission of shock waves. The value of $R_C/R_\kappa$, as a typical example, is taken to be 2.0. The flat surface in the figure represents the state space in which $M^{(I)2}-M^{(II)2}+1 < 0$. The transmitted wave in this region has a form of equation (41) and no wave refraction occurs. At the same value of $R_C/R_\kappa$, the relative amplitudes of reflected and refracted waves shown by equation (39) are displayed in Figs. 12 and 13. On the planes with $M^{(II)}$ being constant, both amplitudes decrease when the value of $M^{(I)}$ ($C^{(II)}$) increases (decreases). On the planes with

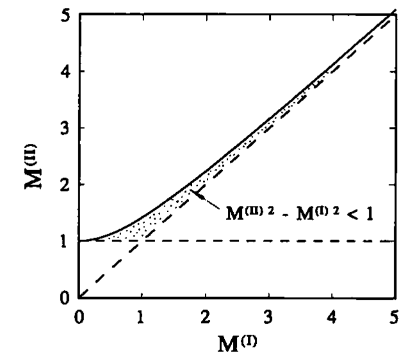


FIG. 10. The physical domain of $M^{(II)2}-M^{(I)2} < 1$ for the existence of refraction wave in the principal direction ($x_2$) of wave propagation.

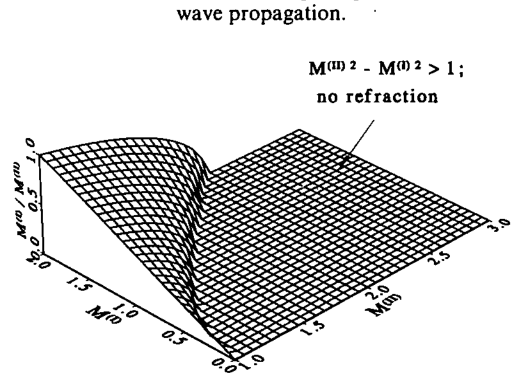


FIG. 11. The envelope for the complete transmission of shock waves. Equation (39) with $R_C/R_\kappa = 2.0$.

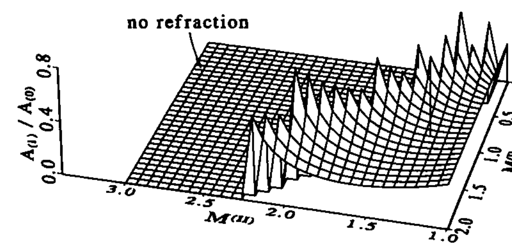


FIG. 12. The relative strength of the reflected shock wave, $A_{(1)}/A_{(0)}$ in equation (38) with $R_C/R_\kappa = 2.0$.

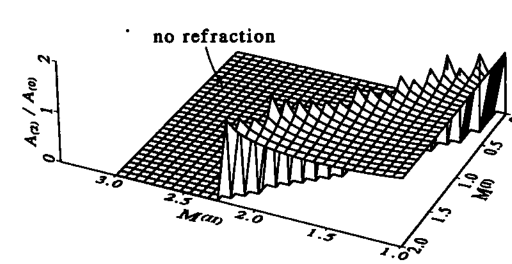


FIG. 13. The relative strength of the refracted shock wave, $A_{(2)}/A_{(0)}$ in equation (38) with $R_C/R_\kappa = 2.0$.

$M^{(I)}$ being constant, on the other hand, relative minima exist at certain values of $M^{(II)}$. By setting the derivatives of $A_{(1)}/A_{(0)}$ and $A_{(2)}/A_{(0)}$ in equation (39) with respect to $M^{(II)}$ be zero, it can be shown that the relative minima of reflected and refracted amplitudes occur at the *same* value of $M^{(II)} = [(M^{(I)2}+1)/2]^{1/2}$.

## CONCLUSION

Reflection and refraction of thermal waves from a surface or an interface between dissimilar materials have been studied by the harmonic analysis. In addition to the harmonic component varying with time sinusoidally, the thermal wave possess a damping with a rate of $1/2\tau$, with $\tau$ being the relaxation time of the solid medium. For an under-damped wave behavior to exist, the wave number $k$ must be greater than the ratio of $C/2\alpha$. In terms of the relaxation time $\tau$, this ratio can also be expressed as $1/2C\tau$. The quantity $C\tau$ is an intrinsic thermal property which, as an extension of the definition for $\tau$, can be defined as the *relaxation distance* in a solid medium. The relaxation distance dominates the transition of thermal waves from an over-damped to an under-damped behavior when subjected to an external frequency excitation [14, 15]. In this work, it is further related to the wave number for an under-damped behavior to exist in the thermal wave propagation. The relaxation distance seems to be an essential quantity in promoting the thermal wave theory to practical applications.

The reflection and refraction patterns of thermal waves from a material interface can be studied, at least in principle, by solving the energy equations in each material layer coupled through the boundary conditions. For the one-dimensional problem, this has been demonstrated by Frankel *et al.* [9]. The convergence of the numerical algorithms for the analytical solutions, however, is very slow in comparison with the parabolic type of equation. From an analytical point of view, this behavior is expected because the energy equation, either $T$- or $\mathbf{q}$-representation in equation (2), is non-self adjoint in nature [24, 27]. It results in a damping factor of $\exp(-t/2\tau)$ which is mainly responsible for the slow convergence. In a recent study by Tzou and Frew [27], for example, reflection of temperature waves from a surface subjected to both temperature- and flux-specified boundary conditions was attempted. The method of Fourier transferm renders an improper integral which has to be computed numerically. Even in determining the reflection angle which appears to be the simplest case in the present approach, the upper bound being infinity has to be increased to the order of $10^6$–$10^7$ for satisfactory convergence. While such a functional approach is capable of determining the amplitude $A_{(0)}$ of the incident wave, combination with the harmonic analysis being proposed in this work facilitates the characterization for reflected and refracted waves. The useful information includes the reflected and refracted angles, the strengths of reflected and refracted waves, and the physical conditions for complete transmission and reflection. Due to slow convergence, these characteristics can hardly be determined by the functional approach alone and the simple relations resulting from the harmonic analysis is illuminating in this sense.

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